Note on the last portion of Chapter 7 of Neukirch–Schmidt–Wingberg

Shun Ishii

Abstract

In the book of Neukirch–Schmidt–Wingberg [NSW08], a nontrivial outer automorphism of the absolute Galois group of $\mathbb{Q}_p$ is constructed whenever $p \neq 2$. We point out that their construction contains a mistake.

Notation

- p is an odd prime.
- $G_{\mathbb{Q}_p} := \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the absolute Galois group of $\mathbb{Q}_p$.
- $V \subset G_{\mathbb{Q}_p}$ is the wild inertia subgroup.
- For $i \geqslant 1$, V^i denotes the i-th term of the lower p-central series.
- $\mathcal{G} := G_{\mathbb{Q}_p}/V = \langle \sigma, \tau \mid \sigma\tau\sigma^{-1} = \tau^p \rangle$ is the maximal tame quotient. We fix a section of $G_{\mathbb{Q}_p} \to \mathcal{G}$, and consider σ, τ as elements of $G_{\mathbb{Q}_p}$.

1 Problem

Theorem 1.1 ([NSW08, the discussion after (7.5.15) Theorem])**.** *$G_{\mathbb{Q}_p}$ has a nontrivial outer isomorphism.*

Let me briefly recall the proof given in *op.cit.* First of all, by Wingberg's result [NSW08, (7.5.15) Theorem], to prove Theorem 1.1, it suffices to construct an automorphism of $G_{\mathbb{Q}_p}/V^3$ satisfying the following two conditions:

1. It is trivial on the maximal tame quotient $\mathcal{G}$, and
2. it is not inner on $G_{\mathbb{Q}_p}/V^2$.

We also introduce Jannsen–Wingberg's theorem [NSW08, (7.5.14) Theorem] on the structure of $G_{\mathbb{Q}_p}$. Let

- $\mathcal{F}_2$ be a free pro-p group of rank two with basis $\{x_0, x_1\}$, and
- $W := \circledast_{\mathcal{G}} \mathcal{F}_2$ is a free pro-p $\mathcal{G}$-operator group of rank two.

Theorem 1.2 ([NSW08, (7.5.14) Theorem])**.** *There exists a section-preserving surjective homomorphism*

$$W \rtimes \mathcal{G} \to V \rtimes \mathcal{G} = G_{\mathbb{Q}_p}$$

such that the kernel is generated (as a normal subgroup) by

$$r := x_0^{-\sigma}\langle x_0, \tau\rangle^g x_1^p [x_1, y_1] \in W.$$

We do not need to know what $\langle x_0, \tau\rangle \in W$, $g \in \mathbb{Z}_p$ and $y_1 \in W$ precisely are. At least $\langle x_0, \tau\rangle$ is an element in the subgroup generated by x_0, σ and τ. By Theorem 1.2, it suffices to construct a suitable automorphism on $W/W^3 \rtimes \mathcal{G}$ which preserves the kernel of $W/W^3 \to V/V^3$. Here, it is easy to construct automorphisms of $W \rtimes \mathcal{G}$, since, for example, for an arbitrary central unit $\varepsilon \in \mathbb{F}_p[[\mathcal{G}]]^\times$, define an endomorphism ψ by

$$\psi(\sigma) = \sigma, \quad \psi(\tau) = \tau, \quad \psi(x_0) = x_0, \quad \text{and} \quad \psi(x_1) = x_1^\varepsilon.$$

via the universality of W. Here we **fix** a lift $x_1^\varepsilon \in W$ of $\varepsilon\bar{x}_1 \in W/W^2$ where $\bar{x}_1 := x_1 \bmod W^2$.[1] Although ψ depends on this lift, ψ always defines an automorphism, since it is so on $W/W^2 \cong \mathbb{F}_p[[\mathcal{G}]]\bar{x}_0 \oplus \mathbb{F}_p[[\mathcal{G}]]\bar{x}_1$ (and since W is a free pro-p group by [NSW08, (4.3.15) Corollary]).

In particular, ψ induces an automorphism on $W/W^3 \rtimes \mathcal{G}$. It is **claimed** that, if ε is carefully chosen, cf. [NSW08, p.422, second paragraph], ψ induces an automorphism on $V/V^3 \rtimes \mathcal{G}$ satisfying the two conditions. However, one can observe the following, which contradicts this **claim**:

Theorem 1.3. *Let $1 \neq \varepsilon \in \mathbb{F}_p[[\mathcal{G}]]$ be an central unit, and ψ be the automorphism of $W/W^3 \rtimes \mathcal{G}$ defined above. Then the kernel of $W/W^3 \to V/V^3$ is **not** preserved by ψ. In other words, $\psi(r)$ is not contained in the kernel.*

Proof. If $\psi(r)r^{-1}$ is contained in the kernel, then we would have

$$\psi(r)r^{-1} = (x_1^\varepsilon)^p[\psi(x_1), \psi(y_1)]x_1^{-p}[x_1, y_1]^{-1} = 1 \bmod V^3.$$

Since the p-th power homomorphism $V/V^2 \to V^2/V^3$ is equivariant with respect to the action of $\mathcal{G}$, we obtain

$$\psi(r)r^{-1} = (x_1^{\varepsilon-1})^p[\psi(x_1), \psi(y_1)][x_1, y_1]^{-1} = 1 \bmod V^3.$$

Observe that the right-hand side is contained in V^2/V^3, and that we have an isomorphism

$$(V/V^2 \wedge V/V^2) \oplus V/V^2 \xrightarrow{\sim} V^2/V^3 \; ; \; (a, b \wedge c) \mapsto a^p + [b, c]$$

since the wild inertia V is a free pro-p group (of countably infinite rank). Hence we must have $x_1^{\varepsilon-1} \in V^2$. However, it is absurd since $\bar{x}_1 \in V/V^2$ generates a free $\mathbb{F}_p[[\mathcal{G}]]$-module (cf. the last paragraph of page 423 in [NSW08]). $\square$

Hence their construction cannot work.

References

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields*, second ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008.

[1] More precisely, in [NSW08, p. 422], x_1^ε is defined by fixing some “ordering”. However, it suffices to take an arbitrary lift of εx_1 in W/W^2 on which $\mathbb{F}_p[[\mathcal{G}]]$ acts in an usual way.